Probability

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Unit 3

Consider the following data:

Probability of a male livebirth during the period 1965–1974

Time period	Number of male livebirths (a)	Total number of livebirths (b)	Empirical probability of a male livebirth (a/b)
1965	1,927,054	3,760,358	.51247
1965–1969	9,219,202	17,989,361	.51248
1965–1974	17,857,857	34,832,051	.51268

Here probabilily means the ratio of the number of "favorable" outcomes to the total number of trials in an actual sequence of experiments.

Definition

Consider a "measure", namely X (male livebirth, temperature, price of a share, etc) or a random experiment (tossing a single six-sided die or tossing a coin). Then the *sample space* is any set of outcomes of interest.

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- X =tossing a coin, $\Omega = \{0, 1\}$.

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Properties

1. *The probability of an event E from the sample space, denoted by $Pr(E)$, satisfies* $0 \leq Pr(E) \leq 1$.
2. *If the outcomes A and B are two evens that cannot both happen at the same time, then*

$$Pr(A \text{ or } B \text{ occurs }) = Pr(A) + Pr(B).$$

Example

Let

$$DBP = \text{ diastolic blood preassure } .$$

Take $A = \{DBP < 90\}$ and $B = \{90 \leq DBP < 95\}$ amd assume $Pr(A) = 0.7$ and $Pr(B) = 0.1$.

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Question

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Since

$$\{DBP < 95\} = \overbrace{\{DBP < 90\}}^{A} \text{ or } \overbrace{\{90 \le DBP < 95\}}^{B} \text{ occurs}$$

and A and B cannot both happen at the same time. Then

$$\begin{aligned} Pr(\{DBP < 95\}) &= Pr(\{DBP < 90\}) + Pr(\{90 \le DBP < 95\}) \\ &= 0.7 + 0.1 = 0.8. \end{aligned}$$

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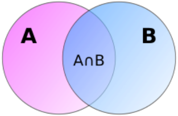

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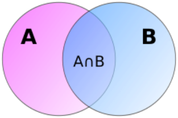

Example

Let $A = \{DBP \geq 90\}$ and $B = \{75 \leq DBP \leq 100\}$. Then

$$A \cap B = \{90 \leq DBP \leq 100\}.$$

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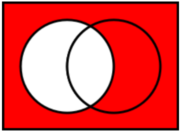

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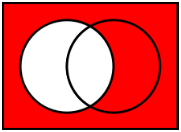

Example

Let $A = \{DBP \geq 90\}$ and $B = \{75 \leq DBP \leq 100\}$. Then

$$\overline{A} = \{DBP < 90\} \text{ and } \overline{B} = \{DPB < 75\} \cup \{DBP > 100\}.$$

Example

Consider as a sample space pairs of DBP measurements of the mother and father within a given family.
Assume that the mother and the father are not genetically related.

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Question

Assume $A = \{$ *mother's DBP* $\geq 95\}$ *and* $B = \{$ *father's DBP* $\geq 95\}$. *Suppose that we know that* $Pr(A) = 0.1$ *and* $Pr(B) = 0.2$. *What can we say about*

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Assume $A = \{$ *mother's DBP* $\geq 95\}$ *and* $B = \{$ *father's DBP* $\geq 95\}$. *Suppose that we know that* $Pr(A) = 0.1$ *and* $Pr(B) = 0.2$. *What can we say about*

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We can say nothing unless we are willing to make certain assumption.

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Example

If we assume that to find a hypertensive mother is independent of to find a hypertensive father. Then

$$Pr(A \cap B) = Pr(\text{ both mother and father are hypertensive })$$
$$= \overbrace{0.1}^{Pr(A)} \times \overbrace{0.2}^{Pr(B)} = 0.02$$

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Example

Let $A = \{$ mother's $DBP \geq 95\}$ and $B = \{$ first-born child's $DBP \geq 80\}$.
Suppose $Pr(A \cap B) = 0.05$, $Pr(A) = 0.1$ and $Pr(B) = 0.2$

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Example

Let $A = \{$ mother's $DBP \geq 95\}$ and $B = \{$ first-born child's $DBP \geq 80\}$. Suppose $Pr(A \cap B) = 0.05$, $Pr(A) = 0.1$ and $Pr(B) = 0.2$

Since

$$Pr(A \cap B) = Pr(\text{ both mother and first-born child's are hypertensive }) = 0.05$$

$$\neq \overbrace{0.1}^{Pr(A)} \times \overbrace{0.2}^{Pr(B)} = 0.02$$

Then A and B are dependent.

Proposition

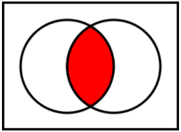
If $A_1, A_2, \ldots, A_n$ are mutually independent events then

$$Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = Pr(A_1) \times Pr(A_2) \times \cdots \times Pr(A_n),$$

Proposition (Addition Law of Probability)

If A and B are any events, then

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$



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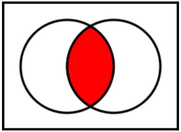

Remark

If the events A and B are mutually exclusives then

$$Pr(A \cap B) = 0, \ \textit{because}\ Pr(A \cup B) = Pr(A) + Pr(B).$$

Example

A dental researcher has been collecting extracted teeth in a jar. The jar contains cuspids, incisors, bicuspids, and molars. Suppose the events are defined by $A = \{\text{incisor}, \text{molar}\}$, $B = \{\text{cuspid}, \text{incisor}\}$, and $C = \{\text{bicuspid}, \text{molar}\}$. He estimated the probabilities of A, B, and C by repeatedly drawing a tooth from the jar, one at a time with a replacement: $Pr(A) = 0.625$, $Pr(B) = 0.375$, $Pr(C) = 0.430$, $Pr(A \cap B) = 0.250$, and $Pr(A \cap C) = 0.375$.

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = 0.625 + 0.375 - 0.250 = 0.750.$$

Here $A \cup B = \{\text{incisor}, \text{molar}, \text{cuspid}\}$.

$$Pr(A \cup C) = Pr(A) + Pr(C) - Pr(A \cap C) = 0.625 + 0.430 - 0.375 = 0.680.$$

Here $A \cup C = \{\text{incisor}, \text{molar}, \text{bicuspid}\}$.

Proposition (Addition Law of Probability for independent events)

If A and B are independent events, then

$$Pr(A \cup B) = Pr(A) + Pr(B) \times [1 - Pr(A)]$$

Proof.

$$\begin{aligned} Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A \cap B) && (1) \\ &= Pr(A) + Pr(B) \times 1 - Pr(A) \times Pr(B) && (2) \\ &= Pr(A) + Pr(B) \times [1 - Pr(A)] && (3) \end{aligned}$$

□

Motivation

Dentists and physicians ask about a patient's lifestyle and individual and family medical histories to ascertain conditional probabilities such as the following:

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Example

- If a patient has consumed a fat and cholesterol- rich diet most of his life, the chances of him developing a myocardial infarction are much greater.
- If an individual is known to have been exposed to an influenza-infected environment, his chance of developing influenza is greater than that of individuals who have not been exposed.

Example

Consider the event

$$A = \{ \text{ a patient has oral infections } \}$$

and

$$B = \{ \text{ a patient develops osteoradionecrosis } \}$$

If $Pr(A) = 0.08$, $Pr(B) = 0.06$, and $Pr(A \cap B) = 0.03$

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Question

What is the probability that a patient which has an oral infection develops osteoradionecrosis ?

Definition

Given two events A and B, $B|A$ denotes the event B conditioned to A.

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Example

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Then

$$B|A = \{ \text{ develops osteoradionecrosis conditional to have an oral infection } \}$$

Definition

The *conditional probability* of B given A is defined by

$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)}$$

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and $B = \{$ a patient develops osteoradionecrosis $\}$ If $Pr(A) = 0.08$, $Pr(B) = 0.06$, and $Pr(A \cap B) = 0.03$ Then

$$P(B|A) = \frac{0.03}{0.08} = 0.375.$$

Proposition

1. *If A and B are independent events, then*

$$Pr(B|A) = Pr(B) = Pr(B|\overline{A}).$$

2. *If A and B are dependent, then*

$$Pr(B|A) \neq Pr(B) \neq Pr(B|\overline{A})$$

and

$$Pr(A \cap B) \neq Pr(A) \times Pr(B).$$

Definition

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Example

Consider the event $A = \{$ a patient has oral infections $\}$ and $B = \{$ a patient develops osteoradionecrosis $\}$ If $Pr(A) = 0.08$, $Pr(B) = 0.06$, and $Pr(A \cup B) = 0.03$ Then

$$RR = \frac{Pr(B|A)}{Pr(B|\overline{A})} = \frac{0.37500}{0.032609} = 11.50.$$

Conclusion

A patien which has an oral infection has 11.50 times probabilities to develops osteoradionecrosis than a patient without an oral infection

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For any events A and B

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Proof.

Note

$$\overbrace{Pr(B \cap A)}^{Pr(B|A)\times Pr(A)} + \overbrace{Pr(B \cap \overline{A})}^{Pr(B|\overline{A})\times Pr(\overline{A})} = Pr(\overbrace{(B \cap A) \cup (B \cap \overline{A})}^{B})$$

□

Example:

$$Pr(\text{ a patient develops osteoradionecrosis }) = Pr(\text{ a patient develops osteoradionecrosis} \mid \text{a patient has oral infections })$$
$$\times Pr(\overbrace{\text{ a patient has oral infections }}^{B_1})$$
$$+Pr(\text{ a patient develops osteoradionecrosis} \mid \text{a patient has no oral infections })$$
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$$\times Pr(\overbrace{\text{ a patient has no oral infections }}^{B_2})$$

Definition

A set of events $B_1, B_2, \ldots, B_n$ is *exahustive* if at least ones of the events must occur.

Proposition (Total-Probability Rule)

Let $B_1, B_2, \ldots, B_n$ mutually exclusive and exahustive events. Then for any event A it follows:

$$Pr(A) = \sum_{i=1}^{n} Pr(A|B_i) \times Pr(B_i).$$

Problem

Tooth mobility is important in the development of a prognosis and vital to treatment planning. Mobility is gauged in millimeters by the motion back and forth in a buccal/lingual direction. It is classified into four categories,

$$\begin{aligned} B_1 &= \text{ no mobility, } Pr(B_1) = 0.05, \\ B_2 &= \text{ class I mobility, } Pr(B_2) = 0.40 \\ B_3 &= \text{ class II mobility, } P(B_3) = 0.30 \text{ and} \\ B_4 &= \text{ class III mobility, } Pr(B_4) = 0.25 \end{aligned}$$

Let A be the event that the patient brushes and flosses at least once every day. If $Pr(A|B_1) = 0.86$, $Pr(A|B_2) = 0.17$, $P(A|B_3) = 0.08$, and $P(A|B_4) = 0.04$, what is the probability that a patient is of class III mobility, knowing that he has been brushing and flossing daily, that is . $Pr(B_4|A)$?

Solution

Note that

$$Pr(B_4|A) = \frac{Pr(B_4 \cap A)}{Pr(A)}$$

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Solution

Note that

$$Pr(B_4|A) = \frac{Pr(B_4 \cap A)}{Pr(A)} = \frac{\overbrace{Pr(A|B_4) \times P(B_4)}^{0.04 \times 0.25}}{Pr(A)}.$$

Since B_1, B_2, B_3 and B_4 are mutually exclusive and exhaustive events. Then, from the total probability rule:

$$\begin{aligned} P(A) =& Pr(A|B_1) \times Pr(B_1) + Pr(A|B_2) \times Pr(B_2) \\ & + Pr(A|B_3) \times Pr(B_3) + Pr(A|B_4) \times Pr(B_4) \\ = & \end{aligned}$$

Theorem

Let $B_1, \cdots, B_n$ be a set of mutually exclusive and exahustive disease states; that is, at least one disease state must occur and non two disease states can occur at the same time. Let A represent the presence of a symptom or set of symtoms. Then

$$Pr(B_i|A) = \frac{Pr(A|B_i) \times Pr(B_i)}{\sum_{i=1}^{n} Pr(A|B_i) \times Pr(B_i)}.$$

Proposition (Generalized Multiplication Law of Probability)

If $A_1, A_2, \ldots, A_n$ are an arbitrary set of events, then

$$\begin{aligned} Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = & Pr(A_1) \times Pr(A_2|A_1) \times Pr(A_3|A_1 \cap A_2) \\ & \times Pr(A_4|A_1 \cap A_2 \cap A_3) \times \cdots \\ & \cdots \times Pr(A_n|A_1 \cap \cdots \cap A_{n-1}). \end{aligned}$$